Soliton and Domain Wall in the Self-Dual $CP(1)$ Model

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Abstract

We perform the dimensional reduction of the nonrelativistic $CP(1)$ model coupled to an Abelian Chern-Simons gauge field in the self-dual limit, and investigate the soliton and domain wall solutions of the emerging 1+1 dimensional self-dual spin system. This system is described by inhomogeneous Landau-Lifshitz system with an extra non-local term. The Hamiltonian is Bogomol'nyi bounded from below and has four adjusting parameters. The Bogomol'nyi equation is described in detail in analogy with the Newtonian equation of motion and its numerical solution is presented.

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1 Introduction

The recent investigation of the dimensional reduction of the Chern-Simons-matter system in 2+1 dimension [1] and its modification uncovered many new features in the soliton physics of 1+1 dimension [2, 3, 4, 5, 6, 7] through the discovery of novel soliton solutions [3]. In particular, the reduction in the self-dual limit is interesting because it can lead to a lineal system which is Bogomol'nyi bounded and therefore, its ground state can be investigated with relative ease. This possibility is demonstrated in Ref. [7], where a self-dual formulation of the non-linear Schrödinger equation (NLSE) in 1+1 dimension is obtained through the dimensional reduction of the Abelian self-dual Chern-Simons theory. The conventional second-order differential equation is replaced by the first-order (non-local) Bogomol'nyi equation which is equivalent to the completely integrable Liouville equation. In the non-Abelian case, similar thing happens but with Toda equation [8].

In this paper, we carry out the dimensional reduction of the self-dual Chern-Simons non-relativistic spin system [9] and study its solutions which saturate the Bogomol'nyi bound. The model in consideration is the non-relativistic $CP(1)$ model minimally coupled to an Abelian Chern-Simons gauge field in a uniform background, which exhibits a rich structure of rotationally symmetric self-dual Bogomol'nyi solitons depending on the various background charges [10]. The dimensionally reduced model turns out to be the self-dual spin system where the inhomogeneous Landau-Lifshitz equation (ILLE) [11] is modified by a non-local interaction. Unlike the self-dual formulation of NLSE [7, 12], this non-local term can neither be replaced by a constant of motion nor be eliminated by a phase redefinition. Rather, it is responsible for a variety of domain walls and magnetic solitons which are not found in the usual spin system [13].

The paper is organized as follows: In section 2, we perform the dimensional reduction of the 2+1 dimensional Chern-Simon spin system to obtain the lineal self-dual non-local ferromagnet model. In section 3, its Bogomol'nyi bound is investigated which leads to the non-local Bogomol'nyi equation. We discuss the features of solitons and domain walls with careful consideration on the range of parameters. Their numerical results are also presented. Section 4 is the conclusion.

2   Dimensional Reduction

In this section, we perform the dimensional reduction of the non-relativistic self-dual Chern-Simons spin system [9, 10] and obtain our lineal model. Let us consider the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= i[(\Psi^\dagger \nabla_0 \Psi) - (\nabla_0 \Psi)^\dagger \Psi] + a_0(\Psi^\dagger \Psi - 1) - \rho_e A_0 \\
&\quad -2|D_i \Psi|^2 - V_{[2]}(\Psi) + \frac{\kappa}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho,
\end{aligned}
\tag{1}
$$

where

$$
\nabla_\mu = \partial_\mu + iA_\mu \frac{\sigma^3}{2}, \ \ D_\mu = \nabla_\mu + ia_\mu.
$$

Ψ is a two component complex field, $\Psi^\dagger = (\Psi_1^*, \Psi_2^*)$, satisfying the $CP(1)$ constraint, $\Psi^\dagger \Psi = 1$, which is imposed through the zeroth component of the auxiliary $U(1)$ gauge field a_μ. A_μ is the $U(1)$ Chern-Simons gauge field, ρ_e is a uniform background charge, and $V_{[2]}(\Psi)$ is a potential energy term given below Eq. (6). The above Lagrangian in $CP(1)$ representation is another expression of the one proposed in Ref. [9, 10] where the Lagrangian is given in the coadjoint representation. The first two terms in the Lagrangian are the familiar canonical symplectic term which can be rewritten as $2\text{Tr}(Kg^{-1}\nabla_0 g)$, $K = i\sigma^3/2, g \in SU(2)$. When

the auxiliary field a_i is eliminated with the help of the $CP(1)$ constraint, the Lagrangian is expressed in terms of the coadjoint orbit variable, $Q = i\Psi\Psi^\dagger - iI/2$. $CP(1)$ representation is used here since it is more convenient for our purpose.

Let us first construct the Bogomol'nyi bound. Using the identity

$$(D_i\Psi)^\dagger(D_i\Psi) = |(D_1 \pm iD_2)\Psi|^2 \pm \epsilon_{ij}\partial_i\left((\Psi^\dagger\tfrac{\sigma^3}{2}\Psi - w)A_j\right)$$
$$\mp i\epsilon_{ij}\partial_i(\Psi^\dagger\partial_j\Psi) \mp \epsilon_{ij}(\partial_i A_j)(\Psi^\dagger\tfrac{\sigma^3}{2}\Psi - w), \tag{2}$$

where ω is a free paameter, we find the Hamiltonian to be

$$H_{[2]} = \int d^2x\left(2|D_i\Psi|^2 + V(\Psi)_{[2]}\right)$$
$$= \int d^2x\left\{2|(D_1 \pm iD_2)\Psi|^2 + V_{[2]}(\Psi) \mp \tfrac{1}{2}F_{ij}(\Psi^\dagger\sigma^3\Psi - 2w)\right\} \mp 4\pi T_{[2]}. \tag{3}$$

F_{ij} is the field strength of the gauge field and the topological charge is given by

$$T_{[2]} = \frac{1}{4\pi}\int d^2x\left\{2i\epsilon_{ij}\partial_i(\Psi^\dagger\partial_j\Psi) + \epsilon_{ij}\partial_i\left[(2w - \Psi^\dagger\sigma^3\Psi)A_j\right]\right\}. \tag{4}$$

Replacing the gauge field strength in terms of matter fields with the help of the Gauss's law constraint

$$\frac{\kappa}{2}\epsilon_{ij}F_{ij} = \Psi^\dagger\sigma^3\Psi + \rho_e, \tag{5}$$

and choosing the form of the potential energy to be

$$V_{[2]}(\Psi) = \pm\frac{1}{\kappa}(Q^3 + \rho_e)(Q^3 - w), \quad Q^3 \equiv \Psi^\dagger\sigma^3\Psi, \tag{6}$$

we have the Hamiltonian with the Bogomol'nyi bound [9, 10]

$$H_{[2]} = 2\int d^2x\left|\left\{\left(\nabla_1 - \tfrac{1}{2}\Psi^\dagger\overleftrightarrow{\nabla}_1\Psi\right) \pm i\left(\nabla_2 - \tfrac{1}{2}\Psi^\dagger\overleftrightarrow{\nabla}_2\Psi\right)\right\}\Psi\right|^2 \mp 4\pi T_{[2]}, \quad \Psi^\dagger\Psi = 1, \tag{7}$$

where a_i has been eliminated by using the equation of motion:

$$a_i = \frac{i}{2}\Psi^\dagger\overleftrightarrow{\partial}_i\Psi - \Psi^\dagger\frac{\sigma^3}{2}\Psi A_i = \frac{i}{2}\Psi^\dagger\overleftrightarrow{\nabla}_i\Psi. \tag{8}$$

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To obtain the Hamiltonian which is dimensionally reduced to the one spatial dimension, we regard Ψ to be independent of y-coordinate. Putting $\frac{\partial}{\partial y} = 0$, and redefining $\Psi \rightarrow e^{-i\int A_1\sigma^3 dx}\Psi$, we have

$$H = 2\int dx \left| \left(\partial_x - iJ_x \mp \frac{1}{2}(\sigma^3 - Q^3)A_2 \right)\Psi \right|^2 \mp 4\pi T\,, \qquad (9)$$

where $J_x = \frac{1}{2i}\Psi^\dagger \overleftrightarrow{\partial_x}\Psi$. T is the one dimensional boundary term which is reduced through the Stokes theorem from $T_{[2]}$ defined in Eq. (4),

$$4\pi T = (Q^3(x) - w) \times A_2(x)|_{x=-\infty} - (Q^3(x) - w) \times A_2(x)|_{x=+\infty}\,. \qquad (10)$$

Solving the Gauss's law constraint, we get the explicit expression for the gauge field A_2,

$$A_2(x) = \frac{1}{2\kappa}\int_{-\infty}^{\infty} K(x-y)(Q^3(y) + \rho_e)dy, \qquad (11)$$

where $K(x)$ is the one-dimensional kernel which solves the Eq. (5):

$$K(x) = \epsilon(x) + \beta. \qquad (12)$$

$\epsilon(x)$ is an odd-step function which is ± 1 depending on the signature of x, and the constant β is to be fixed by a boundary condition. (See the discussion at the end of the next section.)

Now the above Hamiltonian is written in terms of matter field only and it gives the self-dual spin system modified by a non-local interactions. To see this let us expand the Hamiltonian explicitly which yields

$$H = 2\int dx \left[|(\partial_x - iJ_x)\Psi|^2 \pm \frac{1}{2\kappa}(Q^3 + \rho_e)(Q^3 - w) + h(x) \right]\,, \qquad (13)$$

where $h(x)$ is a non-local interaction term,

$$h(x) = \frac{1}{16\kappa^2}(1 - (Q^3)^2)\left| \int K(x-y)(Q^3(y) + \rho_e)dy \right|^2\,.$$

Modulo the nonlocal term $h(x)$, the above Hamiltonian is exactly the same with that of ILLE in the external magnetic field proportional to $\rho_e - w$ in 1+1 dimension. The four parameters, κ, ρ_e, w, and β control the spin direction of the ground state, and the non-local term $h(x)$ supplementing the anisotropic potential energy plays an important role for various solitons and domain walls as we shall see in the next section.

3   Soliton and Domain Wall Solutions

The Hamiltonian in consideration is positive semi-definite and the lower bound is saturated by the non-local Bogomol'nyi equation:

$$\left(\partial_x - iJ_x - \frac{1}{4\kappa}(\sigma^3 - Q^3)\int K(x-y)(Q^3(y) + \rho_e)dy\right)\Psi = 0. \qquad (14)$$

For definiteness, we concentrate on the upper sign (self-dual case) in the Hamiltonian Eq. (9) from now on without loss of generality. (The lower sign (anti-self dual case) can be obtained from the upper sign by switching κ to $-\kappa$.) To solve the above equation, we assume the real ansatz for the wavefunction,

$$\Psi_1 = \frac{1}{\sqrt{1+\rho}}, \quad \Psi_2 = \sqrt{\frac{\rho}{1+\rho}}, \qquad (15)$$

similar to the complex projective coordinate. Then, the current J_x vanishes and Q^3 is expressed as

$$Q^3 = \Psi^\dagger \sigma^3 \Psi = \frac{1-\rho}{1+\rho}. \qquad (16)$$

Putting the above together, we find that the two component equation reduces to a single one

$$(\log \rho)' = -\frac{1}{\kappa}\int_{-\infty}^{\infty} K(x-y)(Q^3(y) + \rho_e)dy, \qquad (17)$$

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which after differentiation on both sides yields

$$(\log \rho)'' = -\frac{2}{\kappa}\left[\left(\rho_e + \frac{1-\rho}{1+\rho}\right)\right]. \tag{18}$$

This equation coincides precisely with the dimensionally reduced vortex equation of Ref. [9, 10]. To interpret this equation as the one dimensional Newtonian equation, we put $\phi = \frac{\ln \rho}{2}$, rescale the coordinate be $x \to x\sqrt{|\kappa|}$, and prepare the equation of motion in the form,

$$\phi'' = -\frac{dW(\phi)}{d\phi}, \quad \phi' = -\int_{-\infty}^{\infty} dy K(x-y)(Q^3(y) + \rho_e), \tag{19}$$

where $W(\phi) = \text{sign}(\kappa)(-\ln\cosh\phi + \rho_e\phi)$, whose form is depicted in Fig. 1 for $\kappa > 0$ and in Fig. 2 for $\kappa < 0$. We may view $W(\phi)$ as an effective potential if we regard x as "time" and ϕ as the position of the hypothetical "particle" with unit mass.
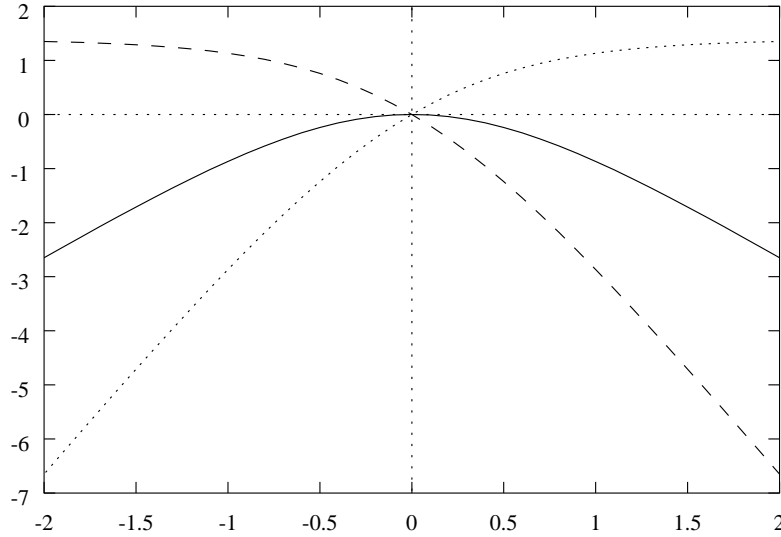

Figure 1: The effective potential: $W(\phi)$ $v.s.$ ϕ when $\kappa > 0$. The solid curve stands for a solution when $\rho_e = 0$, dotted curve for $\rho_e = 1$, and the dashed curve for $\rho_e = -1$.
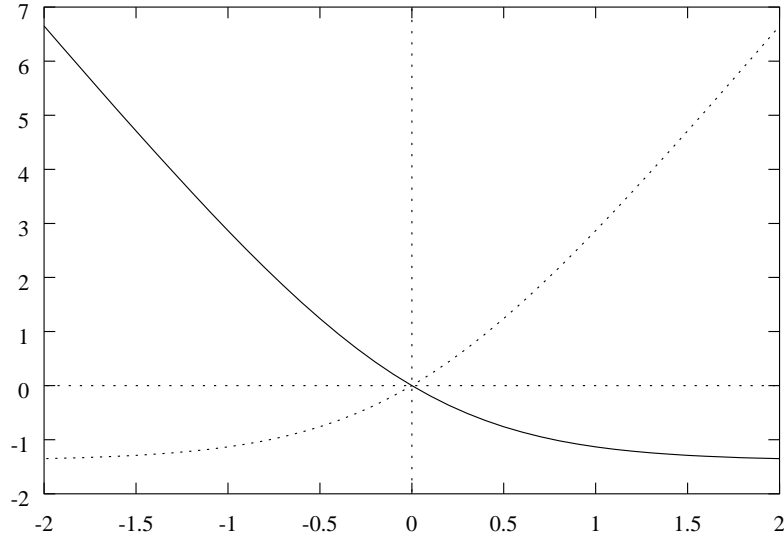
Figure 2: The effective potential: $W(\phi)$ *v.s.* ϕ when $\kappa < 0$. The solid curve stands for a solution when $\rho_e = 1$ and the dotted curve stands for $\rho_e = -1$

Let us consider the solution for the case $\kappa > 0$ first. The effective potential $W(\phi)$ has a bump for $-1 < \rho_e < 1$. Therefore, if a soliton solution exists, then we expect that the soliton solution starts from $\phi = -\infty$ ("far left") at $x = -\infty$ and stops at the top of the hill as $x \to \infty$. This gives the boundary conditions for ϕ,

$$\phi(x = -\infty) = -\infty, \quad \phi'(x = \infty) = 0. \tag{20}$$

It is to be noted that the "initial velocity" $\phi'(x = -\infty)$ should be infinite since the "total energy" is conserved.

If one considers the case in which the "particle" moves from the "far right", then one may change the condition for the starting point as $\phi(x = -\infty) = \infty$. On the other hand, the condition for the final velocity is the same, $\phi'(x = \infty) = 0$. One may equally define the "anti-particle" boundary condition by switching the sign of the "time": the condition at $x = \infty$ is replaced by that at $x = -\infty$ and *vice versa*. Note that with this definition, the "anti-particle" of the particle moving from "far left" and "particle" moving from the "far right" to the top of the hill does not

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coincide. To distinguish this possibility, we will call the condition $\phi(-\infty) = -\infty$ as the "north-pole" condition $(Q^3(-\infty) = 1)$ and the condition $\phi(-\infty) = \infty$ as the "south-pole" condition $(Q^3(-\infty) = -1)$.
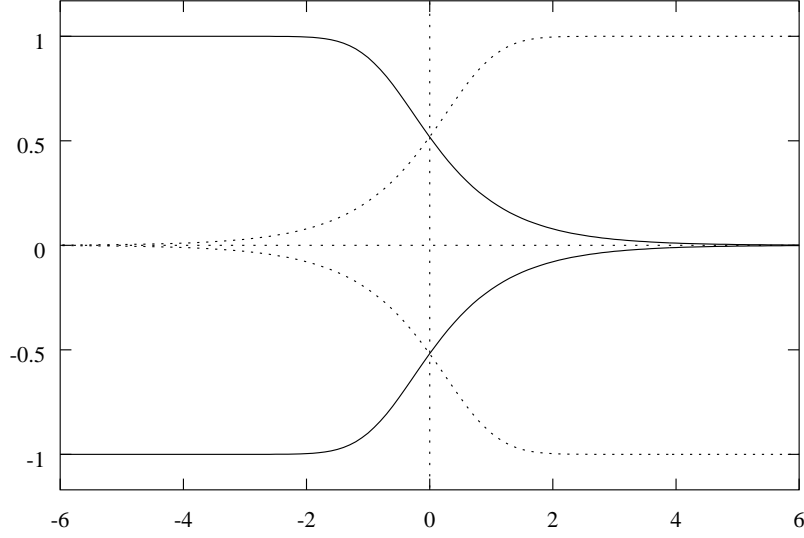

Figure 3: Topological solitons: $Q^3(x)$ $v.s.$ when $\rho_e = 0$. The solid curves stand for solitons and the dotted for an anti-solitons.

We present a numerical solution in Fig. 3 for $\rho_e = 0$. If we define a topological number,

$$q = \frac{1}{2}(Q^3(x = -\infty) - Q^3(x = \infty)),\tag{21}$$

then this soliton solution has the topological number $q = 1/2$ for the north-pole condition since ϕ goes from ∞ to 0 (or Q^3 from 1 to 0). And $q = -1/2$ for the south-pole condition since ϕ goes from ∞ to 0 (or Q^3 from -1 to 0). The anti-soliton has the negative of the topological number of the soliton. There are similar solutions for other values of $|\rho_e| < 1$. The topological number changes smoothly from $\pm\frac{1}{2}$ up to 0 or ±1 as ρ_e varies: $q = \frac{1}{2}(\pm1 + \rho_e)$. When the limiting value $\rho_e = 1(-1)$ reaches, we have a solution corresponding to a lump (anti-lump) which has $q = 1(-1)$. (See Fig. 4 and Fig. 5.) We remark that the boundary condition

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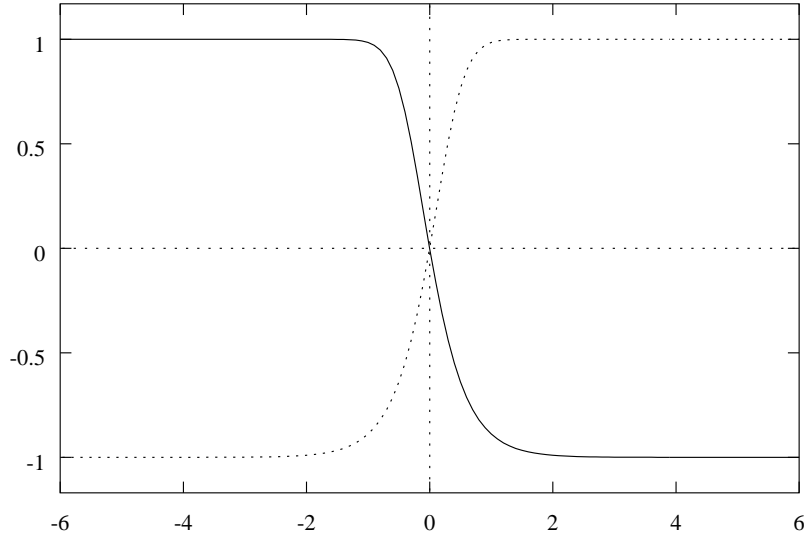for the south-pole lump is the same as that for the north-pole anti-lump.

Figure 4: North-pole solitons: $Q^3(x)$ *v.s.* when $\rho_e = 1$ and $\kappa > 0$. The solid curve stands for a lump and the dotted for an anti-lump.

When $\kappa < 0$, we do not have a topological soliton solution as seen in the $\kappa > 0$ case because the effective potential $W(\phi)$ does not have a bump. (Fig. 2.) Instead, we have a non-topological soliton with $\rho_e = 1$ in which the "particle" starts from $\phi = +\infty$ (south-pole condition), reaches a turning point where it stops, changes the direction, and finally comes back to the original point $\phi = +\infty$ (Q^3 starts from -1 and ends up with -1). (Fig. 6.) For the other case $\rho_e = -1$, we have a dark soliton in which the "particle" starts from the "position" $\phi = -\infty$ (north-pole condition), reaches a point where it stops, changes the direction, and finally comes back to the original "position" $\phi = -\infty$. (Fig. 7.)

Let us turn to the extra boundary term T of Eq. (10). Since T adds to the soliton energy, T has to be finite and be related with a topological quantity if the solitons are to be dynamically stable. We note that T contains the value of $A_2(x)$ at the boundary, $x = \pm\infty$ and that $A_2(x)$ is proportional to the "particle velocity",
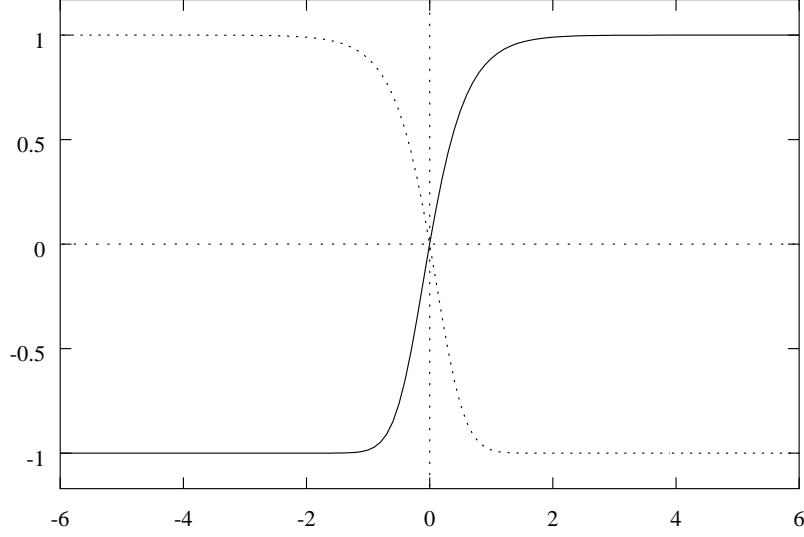
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Figure 5: South-pole solitons: $Q^3(x)$ *v.s.* when $\rho_e = -1$ and $\kappa > 0$. The solid curve stands for a lump and the dotted for an anti-lump.

$\phi'(x)$: $\phi'(x) = -2A_2(x)$. Now, in the case of topological solitons ($\kappa > 0$ and $|\rho_e| \leq 1$), ϕ' cannot be finite simultaneously at both ends ($x = \pm\infty$). This means that T is inevitably ∞ unless there occur some delicate cancellations. The only way to make T finite for the topological solution is to make T vanish by choosing the free parameter w appropriately. For the "particle" case, $A_2(x = \infty) = 0$ and we therefore, must choose $w = Q^3(x = -\infty)$. For the "anti-particle" case, $A_2(x = -\infty) = 0$, and $w = Q^3(x = \infty)$.

For the non-topological soliton ($\kappa < 0$ and $|\rho_e| = 1$), we see that the ϕ' has finite values at both ends ($x = \pm\infty$) with $\phi'(-\infty) = -\phi'(\infty)$ ($Q^3(\infty) = Q^3(-\infty)$), and therefore, T is finite with any value of $|\omega| \leq 1$. The Hamiltonian is bounded by $-4\pi T$ whose value is positive definite when $w \neq Q^3(\infty)$. We can certainly make T vanish by choosing $w = Q^3(\infty)$, whose dynamical stability betrays the non-topological nature of the solution. Nevertheless, the non-topological soliton can exist with the value of w which turns out to be the maximal value for Q^3 in
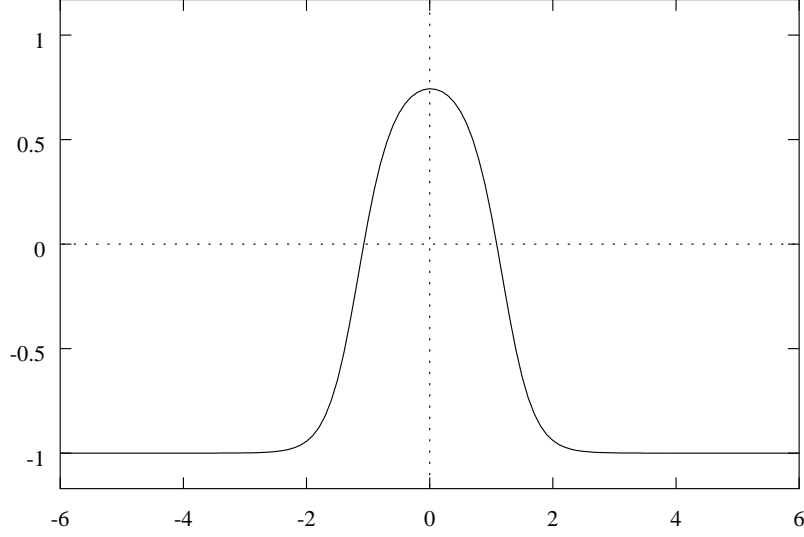
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Figure 6: Non-topological solitons: $Q^3(x)$ $v.s.$ when $\rho_e = 1$ and $\kappa < 0$.

Fig. 6 (minimal value for dark soliton in Fig. 7).

One may suspect that there is a solution for $\kappa > 0$ and $|\rho_e| < 1$, where a "particle" starts to move at $\phi = -\infty$ and jump over the top of the potential energy and goes to the other side, $\phi = +\infty$ (see Fig. 1). It turns out that this configuration is not energetically stable and does not exist. This is because $\phi'(-\infty) = \phi'(\infty)$ and $Q^3(-\infty) \neq Q^3(\infty)$. So $Q^3(-\infty)$ and $Q^3(\infty)$ have to be a pair $(1, -1)$ or $(-1, 1)$. These values, however, make T of Eq. (10) infinite. This is the reason why we do not have the lump soliton for $|\rho_e| < 1$ when $\kappa > 0$.

In addition to the parameter w in the Hamiltonian, Eq. (9) or Eq. (13), the parameter β (in the one-dimensional kernel $K(x)$ in Eq. (19)) is to be fixed properly for the soliton solution to exist. It is fixed by the boundary value of $A_2(x)$. Noting that $A_2(x)$ is proportional to $\phi'(x)$, the condition for A_2 is obtained by the
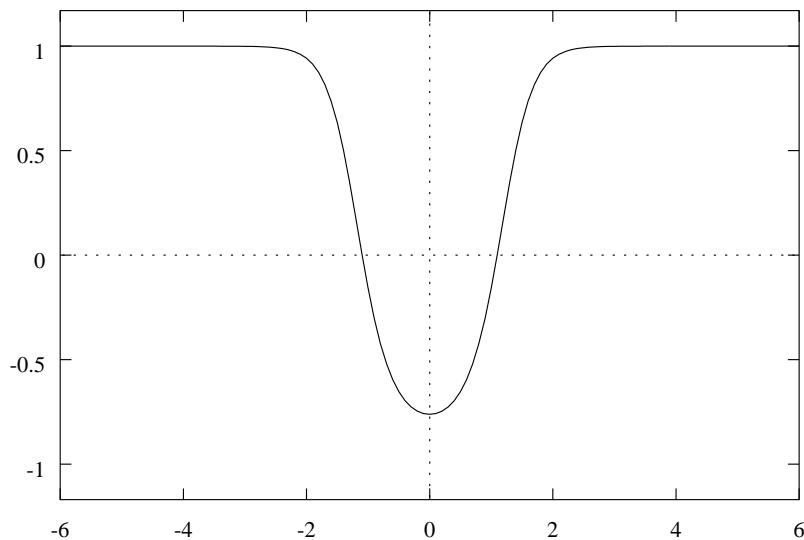
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Figure 7: Dark solitons: $Q^3(x)$ *v.s.* when $\rho_e = -1$ and $\kappa < 0$.

"velocity" at the $\pm$ spatial infinity. According to Eq. (19), we have

$$
\phi'(\infty) = -(1+\beta) \int_{-\infty}^{\infty} dy (Q^3(y) + \rho_e), \quad \phi'(-\infty) = -(-1+\beta) \int_{-\infty}^{\infty} dy (Q^3(y) + \rho_e).
$$
(22)

Since the integration of $Q^3 + \rho_e$ over the whole space need not be zero, we have to choose $\beta = -1$ for the "particle" to stop at the top of the hill and $\beta = 1$ for the "anti-particle". Therefore, our model allows for $\kappa > 0$ either the soliton or the anti-soliton but not the both. For the non-topological case ($\kappa < 0$), β is to be fixed to be null.

4   Conclusion

We carried out a detailed study on the nonlocal self-dual non-relativistic $CP(1)$ system in 1+1 dimension which results from the dimensional reduction of the Abelian self-dual Chern-Simons $CP(1)$ system in 2+1 dimension. We found that

the first order non-local Bogomol'nyi equation yields a second order local equation which offers various soliton solutions.

Our model Hamiltonian, Eq. (9) or Eq. (13) possesses four parameters κ, ρ_e, w, and β, which determine the properties of the soliton and domain wall. The Bogomol'nyi solutions exist only for the specific choice of the parameters. The symmetry of the Hamiltonian under the change of these parameters gives a useful information on the nature of the solutions. For example, self-dual solution with given κ corresponds to the anti-self dual solution with $-\kappa$. Of course, this does not mean that there is one-to-one correspondence between a solution with κ and another solution with $-\kappa$ for the self-dual case or for the anti-self dual case respectively.

The system also possesses the interchnage symmetry between the north and the south-pole, $Q^3 \to -Q^3$. This symmetry is achieved by $\Psi \to \sigma^1 \Psi$ and the simmultaneous change of the signs of two parameters (ρ_e, w). Therefore, a self-dual (anti-self dual) solution with the north-pole condition and (ρ_e, w) guarantees the existence of a self-dual solution (anti-self dual) solution with the south-pole solution and $(-\rho_e, -w)$.

In addition, parity symmetry (or "time-reversal") defined by $x \to -x$, $\Psi \to \pm\Psi$ is broken explicitly in the case when we have non-vanishing values of the parameter β. But the Hamiltonian can still be rendered invariant if the parity transformation is accompanied by the transformation $\beta \to -\beta$. This means that if there is a soliton ("particle") solution with a given value of β, there exist an anti-soliton ("anti-particle") solution corresponding to $-\beta$ which is parity inverted ("time reversed") or *vice versa*.

Finally, we remark that the non-local interaction originates from the gauge field A_2 which is eliminated through the Gauss's law constraint, but is responsible

for the non-gauge type covariant derivative in the Hamiltonian, Eq. (9). This covariant derivative can be replaced by the ordinary derivative if we transform the wavefunction $\Psi \rightarrow \Psi'(\Psi)$ in a suitable way. Then, with $T = 0$, the Hamiltonian becomes free in terms of Ψ'. But this non-linear and non-local transformation does not preserve the symplectic structure of the phase space of Ψ. (Qualitatively, the same is true in the NLSE case [7]). It would be interesting to investigate further the nature of such a non-local and non-linear transformation which renders the Hamiltonian free.

Acknowledgements

References

[1] R. Jackiw and S.-P. Pi, Phys. Rev. Lett. **64**, 2969 (1990); Phys. Rev. D **42**, 3500 (1990); (E) *ibid* **48**, 3929 (1993).

[2] S. J. Benetton Rabello, Phys. Rev. Lett. **76**, 4007 (1996); (E) **77**, 4851 (1996).

[3] U. Aglietti, L. Griguolo, R. Jackiw, S.-Y. Pi and D. Seminara, Phys. Rev. Lett. **77**, 4406 (1996); R. Jackiw, hep-th/9611185.

[4] H. Min and Q-H. Park, Phys. Lett. B **388**, 621 (1996).

[5] H.-C. Kao, K. Lee, and T. Lee, Phys. Rev. D **55**, 6447 (1997).

[6] D. Sen, cond-mat/9612077.

[7] P. Oh and C. Rim, Phys. Lett. B **404**, 89 (1997).

[8] R. Jackiw, hep-th/9705028.

[9] P. Oh and Q-H. Park, hep-th/9612063, Phys. Lett. B **400**, 157 (1997).

[10] Y. Kim, P. Oh and C. Rim, SNUTP/96-20, hep-th/9703193.

[11] L. D. Faddeev and L. A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, Berlin (1987).

[12] I. Andrić, V. Bardek and L. Jonke, hep-th/9507110.

[13] A. M. Kosevich, B. A. Ivanov and A. S. Kovalev, Phys. Rep. **194**, 117 (1990).